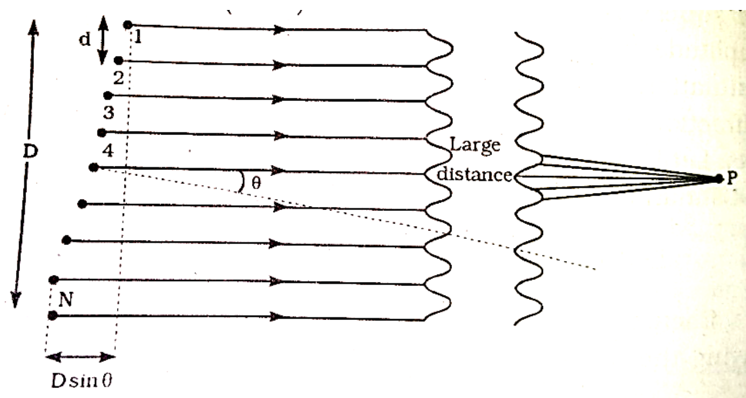
Fraunhoffer Diffraction

Fraunhoffer diffraction:

The diffraction effects in which the source of light and the screen on which the pattern is observed are at far distance from the obstacle is called as Fraunhoffer diffraction.

Composition of N number of SHMs of same amplitude and period having their phases increasing in arithmetic progression



Consider the interference patterns produced due to the superposition of N vibrations of same period, amplitude and phase varying in in arithmetic progression.

Let N be the no. of uniformly space slits, d be the width of each slit and D be the distance between first and last slit.

Then, $D = (n-1)d \ldots..(1)$

Each slit will act as the source of secondary wave having same initial phase

For any j^{th} slit, the displacement at P(point at large distance from slit) after time t is given by

$$y_i = ae^{-i\omega t}e^{i\Phi_j}$$

Where Φ_j is phase of vibration from j^{th} slit at P

Phase difference between two successive slits is given by

$$\alpha = \Phi_{i+1} - \Phi_i = \frac{2\pi}{\lambda}\,(d\sin\theta) \ldots..(2)$$

Similarly

Phase difference between waves between extreme slits is given by

$$\Phi_N - \Phi_1 = \frac{2\pi}{\lambda}\,(D\sin\theta)$$

$$\Phi_N - \Phi_1 = \frac{2\pi}{\lambda}\,(n-1)(d\sin\theta)$$

$$\Phi_N - \Phi_1 = (n-1)\alpha \ldots..(3)$$

By superposition principle of waves, the resultant displacement due to all wave is given by

$$y = y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_N$$

$$y = ae^{-i\omega t}(e^{i\Phi_1} + e^{i\Phi_2} + e^{i\Phi_3} + \ldots\ldots. + e^{i\Phi_N})$$

Multiplying and dividing above equation by $e^{i\Phi_1}$, we get

Prof. Avadhut Manage,
D.M.S. Mandal's Bhaurao Kakatkar College, Belgaum

$$y = ae^{-i\omega t}e^{i\Phi_1}\left(1+\frac{e^{i\Phi_2}}{e^{i\Phi_1}}+\frac{e^{i\Phi_3}}{e^{i\Phi_1}}+\ldots\ldots+\frac{e^{i\Phi_N}}{e^{i\Phi_1}}\right)$$

$$y = ae^{-i\omega t}e^{i\Phi_1}\big(1+e^{i(\Phi_2-\Phi_1)}+e^{i(\Phi_3-\Phi_1)} + e^{i(\Phi_3-\Phi_1)}+\ldots\ldots+e^{i(\Phi_N-\Phi_1)}\big) \ldots.(4)$$

From equation (3), $\Phi_N - \Phi_1 = (n-1)\alpha$, we get

$$y = ae^{-i\omega t}e^{i\Phi_1}\big(1+e^{i\alpha}+e^{2i\alpha}+e^{3i\alpha}+\ldots\ldots+e^{i(N-1)\alpha}\big)$$

But $\big(1+e^{i\alpha}+e^{2i\alpha}+e^{3i\alpha}+\ldots\ldots+e^{i(N-1)\alpha}\big) = \frac{e^{iN\alpha}-1}{e^{i\alpha}-1}$

$$\therefore y = ae^{-i\omega t}e^{i\Phi_1}\frac{e^{iN\alpha}-1}{e^{i\alpha}-1}$$

$$y = ae^{-i\omega t}e^{i\Phi_1}\frac{e^{\frac{iN\alpha}{2}}}{e^{\frac{i\alpha}{2}}} X \frac{e^{\frac{iN\alpha}{2}}-e^{-\frac{iN\alpha}{2}}}{e^{\frac{i\alpha}{2}}-e^{-\frac{i\alpha}{2}}}$$

$$y = ae^{-i\omega t}e^{i\Phi_1}e^{\frac{i\alpha(N-1)}{2}} X \frac{e^{\frac{iN\alpha}{2}}-e^{-\frac{iN\alpha}{2}}}{e^{\frac{i\alpha}{2}}-e^{-\frac{i\alpha}{2}}}$$

$$\because \frac{e^{\frac{iN\alpha}{2}}-e^{-\frac{iN\alpha}{2}}}{e^{\frac{i\alpha}{2}}-e^{-\frac{i\alpha}{2}}} = \frac{\sin\frac{N\alpha}{2}}{\sin\frac{i\alpha}{2}}$$

$$\therefore y = ae^{-i\omega t}e^{i\Phi_1}e^{\frac{i\alpha(N-1)}{2}} X \frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}}$$

$$\therefore y = ae^{-i\omega t}e^{i\left(\Phi_1+\frac{\alpha(N-1)}{2}\right)} X \frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}}$$

But $\Phi_1 + \frac{\alpha(N-1)}{2} = \Phi_1 + \frac{1}{2}(\Phi_N - \Phi_1) = \frac{1}{2}(\Phi_N + \Phi_1) = \Phi_{Av}$

$$\because y = ae^{-i\omega t}e^{i\Phi_{Av}} X \frac{\sin\frac{N\alpha}{2}}{\sin\frac{i\alpha}{2}}$$

Taking real part , the real part of y is given by

$$y = a\frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}}\cos(\omega t - \Phi_{Av})$$

Substituting α in above equation, we get

$$y = a\frac{\sin\frac{N}{2}\left(\frac{2\pi}{\lambda}\ d\sin\theta\right)}{\sin\frac{1}{2}\left(\frac{2\pi}{\lambda}\ d\sin\theta\right)}\cos(\omega t - \Phi_{Av})$$

$$y = a\frac{\sin\left(\frac{N\pi}{\lambda}\ d\sin\theta\right)}{\sin\left(\frac{\pi}{\lambda}\ d\sin\theta\right)}\cos(\omega t - \Phi_{Av})$$

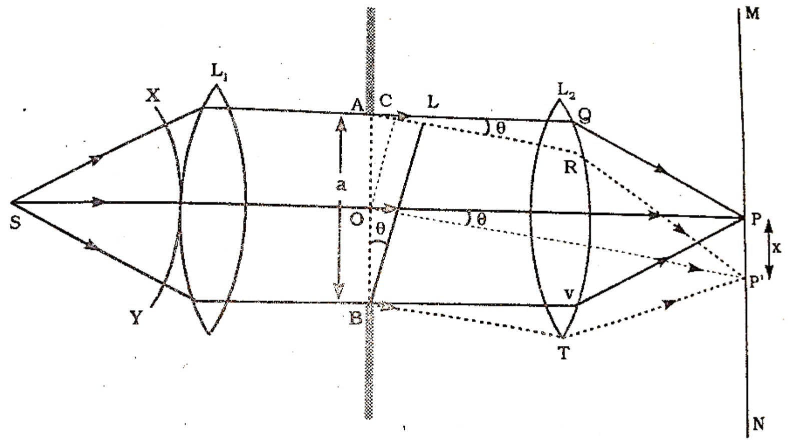
Substituting $d = \frac{D}{N-1}$ in above equation, we get

$$y = a\frac{\sin\left(\frac{N\pi}{N-1}\ \frac{D\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi}{N-1}\ \frac{D\sin\theta}{\lambda}\right)}\ \cos(\omega t - \Phi_{Av})$$

This formula is valid for $N \geq 2$

Prof. Avadhut Manage,
D.M.S. Mandal's Bhaurao Kakatkar College, Belgaum

Fraunhoffer diffraction



Consider a Source of light S produces divergent beam of light of wavelength λ. Those all ray are rendered to parallel beam of light or a plane wave front using first convex lens L_1 and it is incident on a narrow slit AB of width a. The pattern due to diffraction at the slit is focused on screen with the help of second convex lens L_2.

The diffraction pattern on the screen can be explained using Huygens principle as follows. Consider the plane wavefront AB, each point on this wavefront send out the secondary wavelets in all directions. The secondary waves from all the points on the wavefront AB travel the same distance in reaching the point P on the screen. Hence the path difference between them is zero. These secondary waves superpose with each other and produce maximum intensity at P. Thus P is location of **central maximum** on the screen.

Now consider the secondary waves travelling towards P' making an angle θ with OP (O is the centre of the slit AB) and at distance x from centre P. The intensity at P' depends on the path difference between the secondary wavelets emitted from the A and B. BL is perpendicular drawn from B to the ray diffracted from A. Then path difference between the secondary waves from A and B reaching the point P' is equal to AL.

From $\Delta^{le} ABL$

$$\sin\theta = \frac{AL}{AB} = \frac{AL}{a}$$

$$AL = a\sin\theta$$

For n^{th} minimum,

$$a\sin\theta_n = n\lambda$$

$$\sin\theta_n = \frac{n\lambda}{a}$$

Where θ_n is angle of diffraction for n^{th} minimum.

For n^{th} maximum,

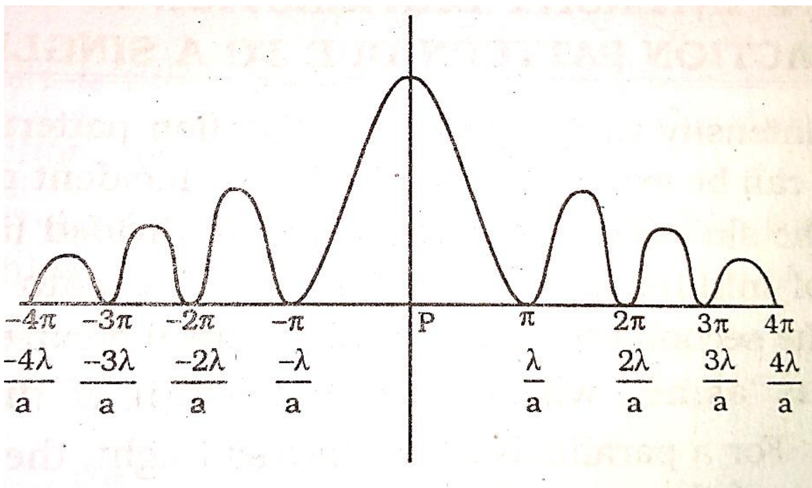
$$a\sin\theta_n = (2n+1)\frac{\lambda}{2}$$

$$\sin\theta_n = \frac{(2n+1)\lambda}{2a}$$

Where θ_n is angle of diffraction for n^{th} minimum.

The diffraction pattern obtained on the screen consists of central bright band called as **central maximum** and alternate dark and bright bands of decreasing intensity on either side called as **secondary minimum** and **secondary maximum** respectively.

The intensity variation on the screen is represented in below figure.



If the lens L_2 and L_1 is very near to the slit and the screen is far away, then

$$\sin\theta = \frac{x}{f}$$

(Where f is the focal length of lens L_2)

For first minimum,

$$\frac{\lambda}{a} = \frac{x}{f}$$

$$x = \frac{f\lambda}{a}$$

Where x is distance of first minimum from centre.

Therefore width of central maximum is $2x$

$$ie, 2x = \frac{2f\lambda}{a}$$

Intensity of maxima is given by

$$I = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2$$

Where $\alpha = \frac{\pi}{\lambda} a \sin\theta$

Diffraction Grating

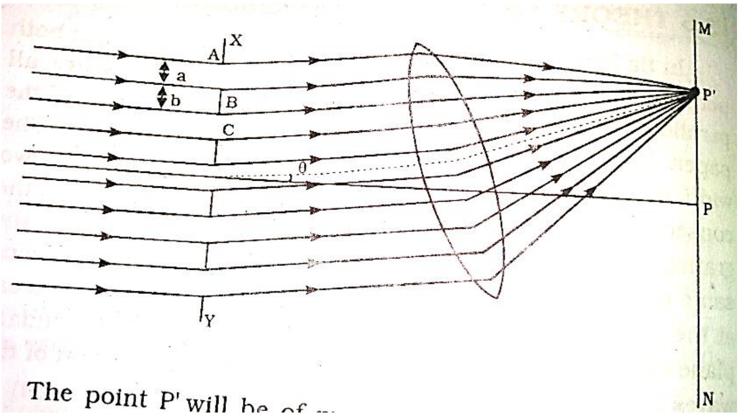
A diffraction grating is an arrangement of large number of parallel slits of equal width separated from each other by equal space. For practical applications, gratings generally have ridges or rulings on their surface rather than dark lines. Such gratings can be either transmissive or reflective. Grating contains 6,000 lines/cm to 10,000 lines/cm

Grating are used to study the spectrum of light and to determine the wavelength of light.

If a is the width of transparent part (slit) and b is width of opaque part (line drawn) then $(a + b)$ is called grating element. If there N lines per meter in grating then $(a + b) = \frac{1}{N}$

Prof. Avadhut Manage,
D.M.S. Mandal's Bhaurao Kakatkar College, Belgaum

Theory of plane transmission of grating



Let a is the width of transparent part (slit) and b is width of opaque part (line drawn) then $(a+b)$ is called grating element. If there N lines per meter in grating then $(a+b) \ = \frac{1}{N}$

The path difference between the secondary waves starting from A to C reaching the point P' is given by

$$\Delta = AC \ \sin\theta = (a+b)\sin\theta$$

For n^{th} maximum,

$$(a+b)\sin\theta_n = n\lambda$$

Where θ_n is angle of diffraction for n^{th} maximum.

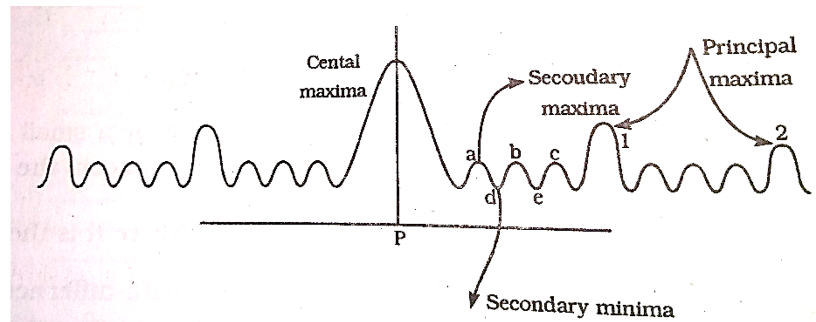
If the incident light consists of more than one wavelength. Then diffraction for different wavelength is different. Let λ and $d\lambda$ be the two nearby wavelengths present in incident light and θ and $(\theta + d\theta)$ be the angles of diffraction corresponds to these wavelengths

Then the n^{th} maximum is given by

$$(a+b)\sin(\theta_n + d\theta) = n(\lambda + d\lambda)$$

Where $n = 1$, $n = 2$ .... Gives first order, second order spectrum respectively

When white light is used, the diffraction pattern consists of a white central bright maximum and on both sides of this maximum a spectrum of a different colours are obtained on the screen as shown in the figure.



Suppose the angle of diffraction is increased by an amount $d\theta$, such the path difference between the secondary waves from A to C increased by $\frac{\lambda}{N}$

Then the path difference between extreme secondary waves is given by

$$\frac{\lambda}{N} X \ N = \lambda$$

Hence for any upper half wave there will corresponding lower half wave having path difference $\frac{\lambda}{2}$, which gives rise to minima

Prof. Avadhut Manage,
D.M.S. Mandal's Bhaurao Kakatkar College, Belgaum

Hence $\frac{\lambda}{N}, \frac{2\lambda}{N}, \frac{3\lambda}{N} \ldots\ldots \frac{(N-1)\lambda}{N}$ path difference between secondary waves from neighbor slits gives $1, 2, 3, \ldots\ldots, (N-1)$ minima respectively.

Similarly the angle of diffraction is increased by an amount $d\theta$, such the path difference between the secondary waves from A to C increased by $\frac{(2n+1)\lambda}{2N}$

Then the path difference between extreme secondary waves is given by

$$\frac{(2n+1)\lambda}{2N} \; X \; N = (2n+1)\frac{\lambda}{2}$$

which gives rise to maxima

Hence $\frac{\lambda}{2N}, \frac{3\lambda}{2N}, \frac{5\lambda}{N} \ldots\ldots \frac{(N-2)\lambda}{2N}$ path diffence between secondary waves from neighbor slits gives $1, 2, 3, \ldots\ldots, (N-2)$ maxima respectively.

Dispersive power

Dispersive power of grating is defined as the ratio of difference in the angle of diffraction per unit change in the wavelength.

$$ie. \frac{d\theta}{d\lambda}$$

The diffraction of n^{th} order maxima is for wavelength λ is given by

$$(a+b)\sin\theta = n\lambda$$

Differentiating with respect to θ, we get

$$(a+b)\cos\theta = n\,\frac{d\lambda}{d\theta}$$

$$\therefore \frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta}$$

$$\because (a+b) = \frac{1}{N}$$

$$\therefore Dispersive\ power, \frac{d\theta}{d\lambda} = \frac{nN}{\cos\theta}$$

If the distance between the lens and screen in x ,

We have,

$$\sin\theta = \frac{x}{f}$$

Or

$$\sin d\theta = \frac{dx}{f}$$

For small $d\theta$, $\sin d\theta \approx d\theta$

$$d\theta = \frac{dx}{f}$$

$$\therefore \frac{d\theta}{d\lambda} = \frac{dx}{f d\lambda}$$

$$\therefore \frac{dx}{d\lambda} = f\frac{d\theta}{d\lambda} == \frac{fnN}{\cos\theta}$$

Prof. Avadhut Manage,
D.M.S. Mandal's Bhaurao Kakatkar College, Belgaum

Resolving power

Resolving power of an optical instrument is its ability to produce distinctly separate images of two closely lying point objects.

It can be defined as the ratio of wavelength λ to the smallest change in the wavelength .

$$\therefore R.P = \frac{\lambda}{d\lambda}$$

Resolving power of grating

We know, for wavelength λ and For n^{th} maximum,

$$(a+b)\sin\theta_n = n\lambda \quad \dots\dots(1)$$

Where θ_n is angle of diffraction for n^{th} maximum.

for wavelength $\lambda + d\lambda$ and For n^{th} maximum,

$$(a+b)\sin(\theta_n + d\theta) = n(\lambda + d\lambda) \dots..(2)$$

These two lines will appear to just separated (resolved) if the angle of diffraction $(\theta_n + d\theta)$ also corresponds to the direction of the first secondary minimum after n^{th} primary maximum at point P (corresponding to the wavelength λ). This is possible if the extra path difference introduced is $\frac{\lambda}{N}$. Then equation (1) becomes

$$\therefore\ (a+b)\sin(\theta_n + d\theta) = n\lambda + \frac{\lambda}{N} \dots\dots(3)$$

Since when extra path difference introduced is $\frac{\lambda}{N}$ is introduced angle of diffraction also changes from θ_n to $\theta_n + d\theta$

From eq^n (3) and (2) , we get

$$\therefore\ (a+b)\sin(\theta_n + d\theta) = n\lambda + \frac{\lambda}{N} = n(\lambda + d\lambda)$$

$$\therefore\ n\lambda + \frac{\lambda}{N} = n\lambda + nd\lambda$$

$$\therefore\ \frac{\lambda}{N} = nd\lambda$$

$$\therefore\ \frac{\lambda}{d\lambda} = nN$$

$$\therefore Resolving\ power = nN$$

Prof. Avadhut Manage,
D.M.S. Mandal's Bhaurao Kakatkar College, Belgaum